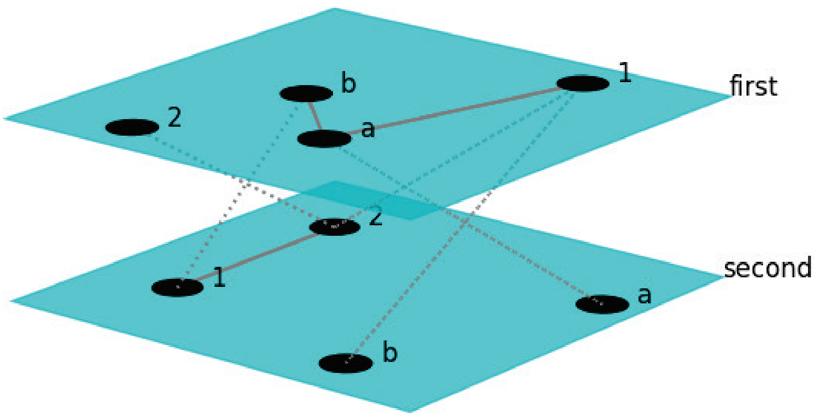
Multilayer Networks



Hiroki Sayama
sayama@binghamton.edu

Multilayer networks

- Networks that are composed of multiple layers (tautological…)
- A number of different names:
  - Multilayer networks
  - Multiplex networks
  - Interconnected networks
  - Interdependent networks
  - Networks of networks

  etc…

Heads up

- This is even newer than temporal network research
- Tools for multilayer network research still under active development
  - But there are some tools now available ☺
- More math/theory-driven, and involve more dynamical models, than temporal network research

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Multilayer networks

MIKKO KIVELÄ
Oxford Centre for Industrial and Applied Mathematics, Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

ALEX ARENAS
Departament d'Enginyeria Informática i Matemátiques, Universitat Rovira I Virgili, 43007 Tarragona, Spain

MARC BARTHELEMY
Institut de Physique Théorique, CEA, CNRS-URA 2306, F-91191, Gif-sur-Yvette, France and Centre d'Analyse et de Mathématiques Sociales, EHESS, 190-198 avenue de France, 75244 Paris, France

JAMES P. GLEESON
MACSI, Department of Mathematics & Statistics, University of Limerick, Limerick, Ireland

YAMIR MORENO
Institute for Biocomputation and Physics of Complex Systems (BIFI), University of Zaragoza, Zaragoza 50018, Spain and Department of Theoretical Physics, University of Zaragoza, Zaragoza 50009, Spain

AND

MASON A. PORTER[†]
Oxford Centre for Industrial and Applied Mathematics, Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK and CABDyN Complexity Centre, University of Oxford, Oxford OX1 1HP, UK
[†]Corresponding author. Email: porterm@maths.ox.ac.uk

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In most natural and engineered systems, a set of entities interact with each other in complicated patterns that can encompass multiple types of relationships, change in time and include other types of complications. Such systems include multiple subsystems and layers of connectivity, and it is important to take such 'multilayer' features into account to try to improve our understanding of complex systems. Consequently, it is necessary to generalize 'traditional' network theory by developing (and validating) a framework and associated tools to study multilayer systems in a comprehensive fashion. The origins of such efforts date back several decades and arose in multiple disciplines, and now the study of multilayer networks has become one of the most important directions in network science. In this paper, we discuss the history of multilayer networks (and related concepts) and review the exploding body of work on such networks. To unify the disparate terminology in the large body of recent work, we discuss a general framework for multilayer networks, construct a dictionary of terminology to relate the numerous existing concepts to each other and provide a thorough discussion that compares, contrasts and translates between related notions such as multilayer networks, multiplex networks, interdependent networks, networks of networks and many others. We also survey and discuss existing data sets that can be represented as

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The structure and dynamics of multilayer networks

S. Boccaletti[a,b,*], G. Bianconi[c], R. Criado[d,e], C.I. del Genio[f,g,h], J. Gómez-Gardeñes[i], M. Romance[d,e], I. Sendiña-Nadal[j,e], Z. Wang[k,l], M. Zanin[m,n]

[a] *CNR - Institute of Complex Systems, Via Madonna del Piano, 10, 50019 Sesto Fiorentino, Florence, Italy*
[b] *The Italian Embassy in Israel, 25 Hamered st., 68125 Tel Aviv, Israel*
[c] *School of Mathematical Sciences, Queen Mary University of London, London, United Kingdom*
[d] *Departamento de Matemática Aplicada, Universidad Rey Juan Carlos, 28933 Móstoles, Madrid, Spain*
[e] *Center for Biomedical Technology, Universidad Politécnica de Madrid, 28223 Pozuelo de Alarcón, Madrid, Spain*
[f] *Warwick Mathematics Institute, University of Warwick, Gibbet Hill Road, Coventry CV4 7AL, United Kingdom*
[g] *Centre for Complexity Science, University of Warwick, Gibbet Hill Road, Coventry CV4 7AL, United Kingdom*
[h] *Warwick Infectious Disease Epidemiology Research (WIDER) Centre, University of Warwick, Gibbet Hill Road, Coventry CV4 7AL, United Kingdom*
[i] *Institute for Biocomputation and Physics of Complex Systems, University of Zaragoza, Zaragoza, Spain*
[j] *Complex Systems Group, Universidad Rey Juan Carlos, 28933 Móstoles, Madrid, Spain*
[k] *Department of Physics, Hong Kong Baptist University, Kowloon Tong, Hong Kong Special Administrative Region*
[l] *Center for Nonlinear Studies, Beijing–Hong Kong–Singapore Joint Center for Nonlinear and Complex Systems (Hong Kong) and Institute of Computational and Theoretical Studies, Hong Kong Baptist University, Kowloon Tong, Hong Kong Special Administrative Region*
[m] *Innaxis Foundation & Research Institute, José Ortega y Gasset 20, 28006 Madrid, Spain*
[n] *Faculdade de Ciências e Tecnologia, Departamento de Engenharia Electrotécnica, Universidade Nova de Lisboa, 2829-516 Caparica, Portugal*

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ABSTRACT

In the past years, network theory has successfully characterized the interaction among the constituents of a variety of complex systems, ranging from biological to technological, and social systems. However, up until recently, attention was almost exclusively given to networks in which all components were treated on equivalent footing, while neglecting all the extra information about the temporal- or context-related properties of the interactions under study. Only in the last years, taking advantage of the enhanced resolution in real data sets, network scientists have directed their interest to the multiplex character of real-world systems, and explicitly considered the time-varying and multilayer nature of networks. We offer here a comprehensive review on both structural and dynamical organization of graphs made of diverse relationships (layers) between its constituents, and cover several relevant issues, from a full redefinition of the basic structural measures, to understanding how the multilayer nature of the network affects processes and dynamics.

* Corresponding author.
E-mail address: stefano.boccaletti@gmail.com (S. Boccaletti).

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Examples

- Social network that involve different types of connections
- Network of airports connected by different air carriers
- Multiple infrastructures of a nation that are connected to each other
- Species interaction patterns that involve distinct layers of habitat

Fundamentals of Multilayer Networks

General representation

- To represent a multilayer network, define the following function:

$$a(i, \alpha, j, \beta) = 0 \text{ or } 1 \text{ (or weight)}$$

- $a(i, \alpha, j, \beta) = 1$ if node i in layer α is connected to node j in layer β, otherwise 0

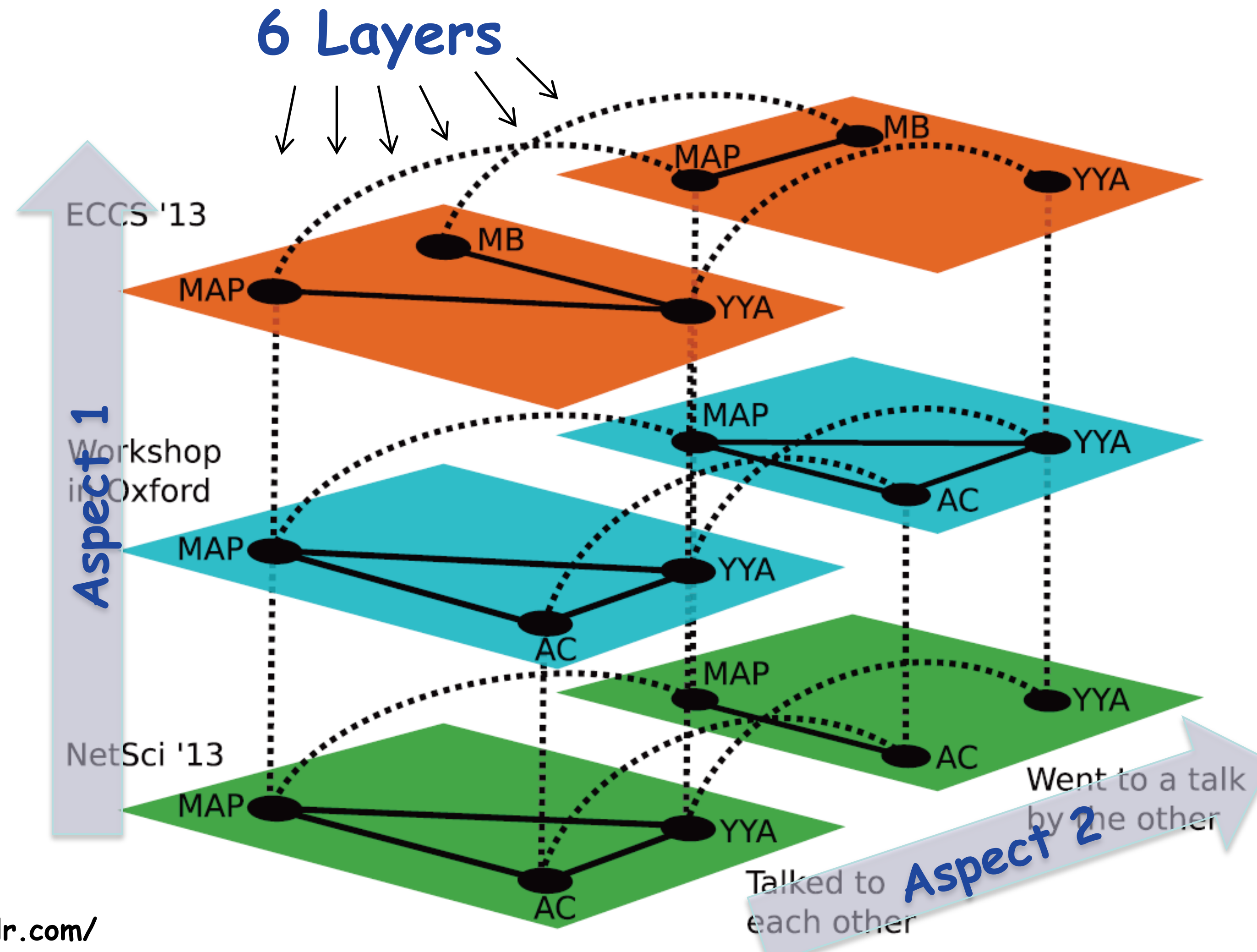
  (This can be represented mathematically by an adjacency tensor)

Aspects

- Layers α, β can be mapped to a space made of multiple "**aspects**"
  - Time, location, type of connection, etc.
  - This means α, β can be vectors
- Aspects help organize the relationships between different layers

Example: Zachary's Karate Club Club (ZKCC) network

http://networkkarate.tumblr.com/



Source: Kivelä et al. (2014)

Exercise

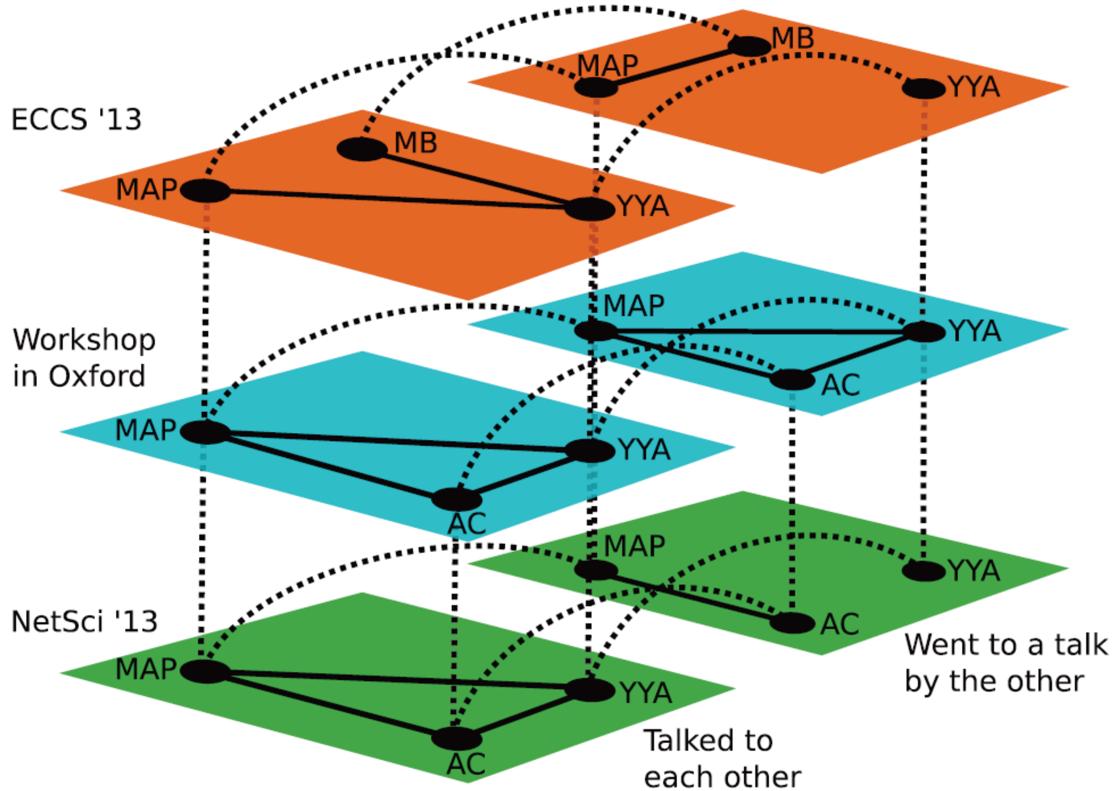
- Write down the values of the adjacency function a(i, α, j, β) for some edges in the previous ZKCC network
- Represent the same information in an edge list

Different classes of edges

- Intra-layer edge $[(i, \alpha), (j, \alpha)]$
- Inter-layer edge $[(i, \alpha), (j, \beta)]$ $\alpha \neq \beta$
  - In particular:
    Coupling edge $[(i, \alpha), (i, \beta)]$ $\alpha \neq \beta$

Exercise

- Identify (a) intra-layer edges, (b) inter-layer edges and (c) coupling edges in this multilayer network

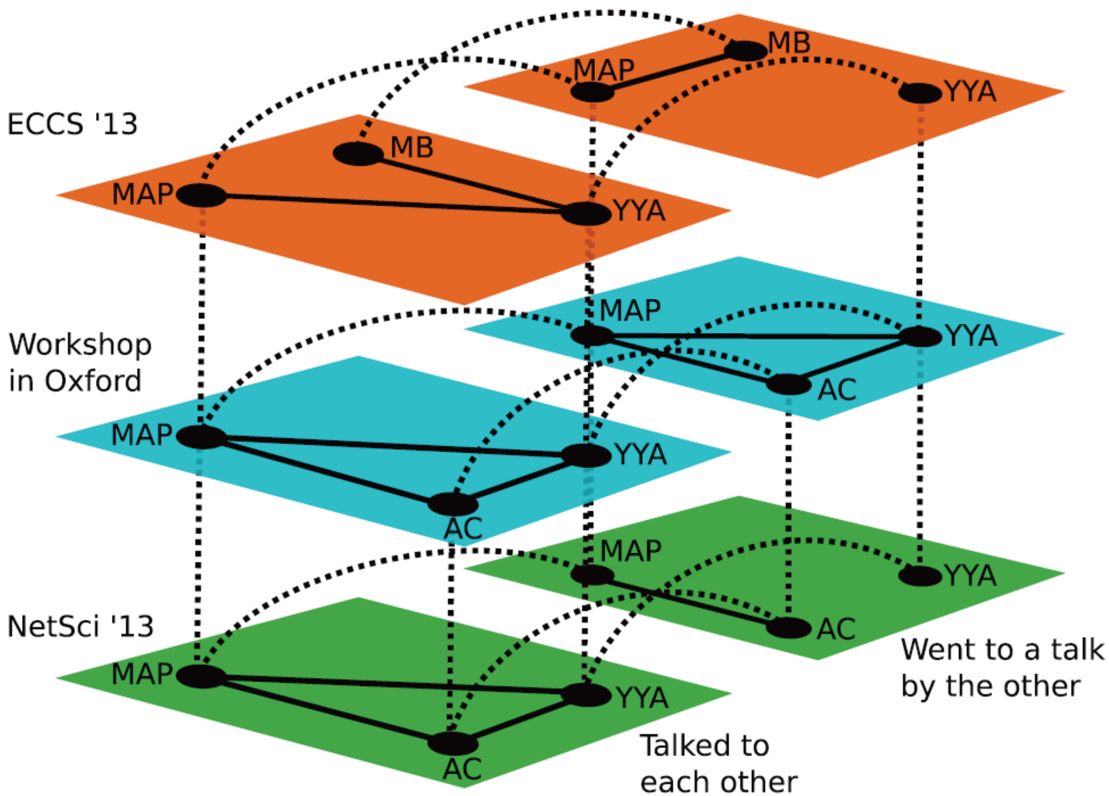


Properties of multilayer networks

- Do all the layers contain the same set of nodes? → node-aligned
- Or, does each node appear only in one layer? → layer-disjoint
- Are all the inter-layer edges coupling ones? → diagonally coupled
  - Are the diagonal coupling edges independent of nodes? → layer-coupled

Exercise

- **Is this network (a) node-aligned or layer-disjoint? (b) diagonally coupled? (c) layer-coupled?**



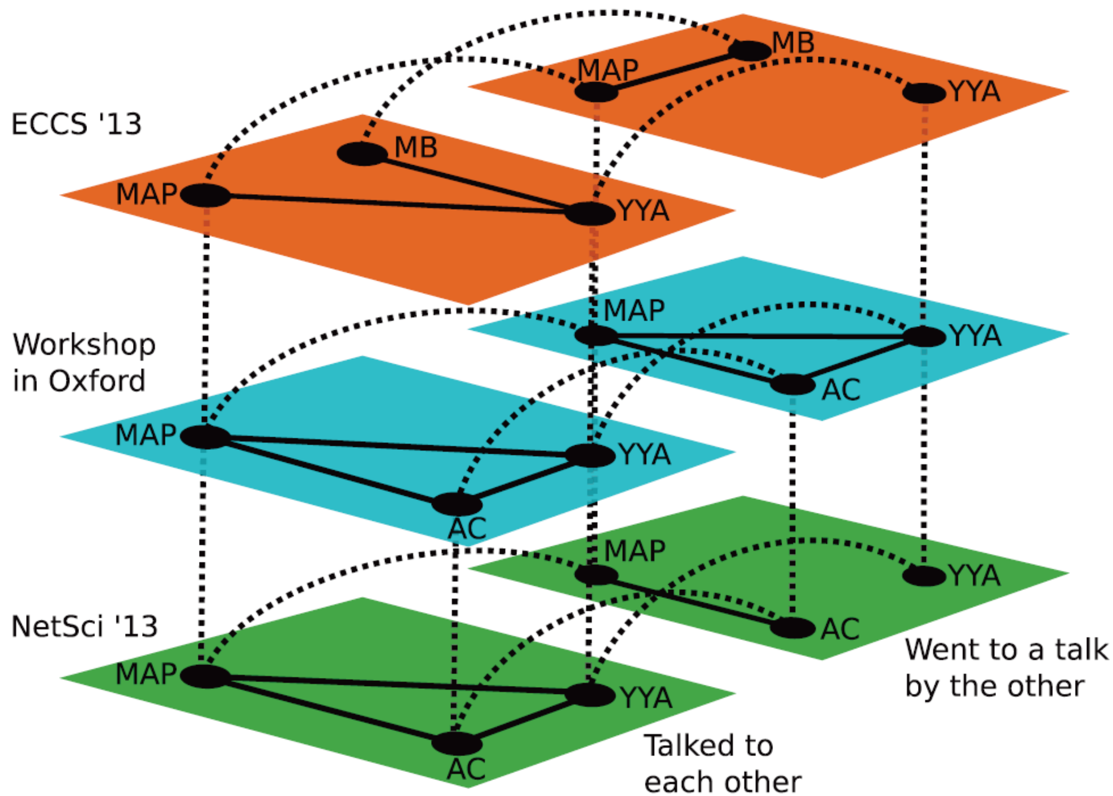
Different kinds of diagonal coupling along an aspect

- Do the diagonal edges connect a node to its counterparts in all other layers along an aspect? → categorical

- Or, do the diagonal edges connect a node to its counterparts only in "nearby" layers along an aspect? → ordinal

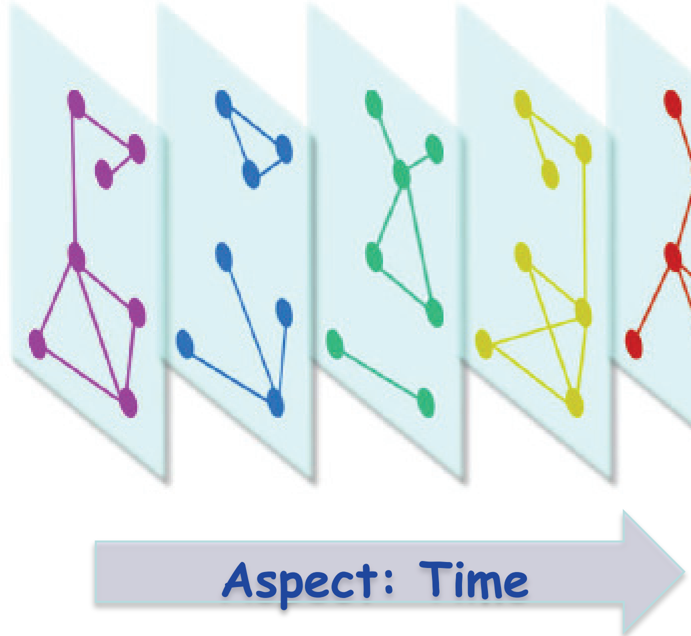
Exercise

- Does each of the two aspects of this network (a) categorical or (b) ordinal?



Temporal networks as multilayer networks

- Temporal networks can also be considered a special case of multilayer networks that have only one ordinal aspect: time



Source: De Domenico et al. (2013)

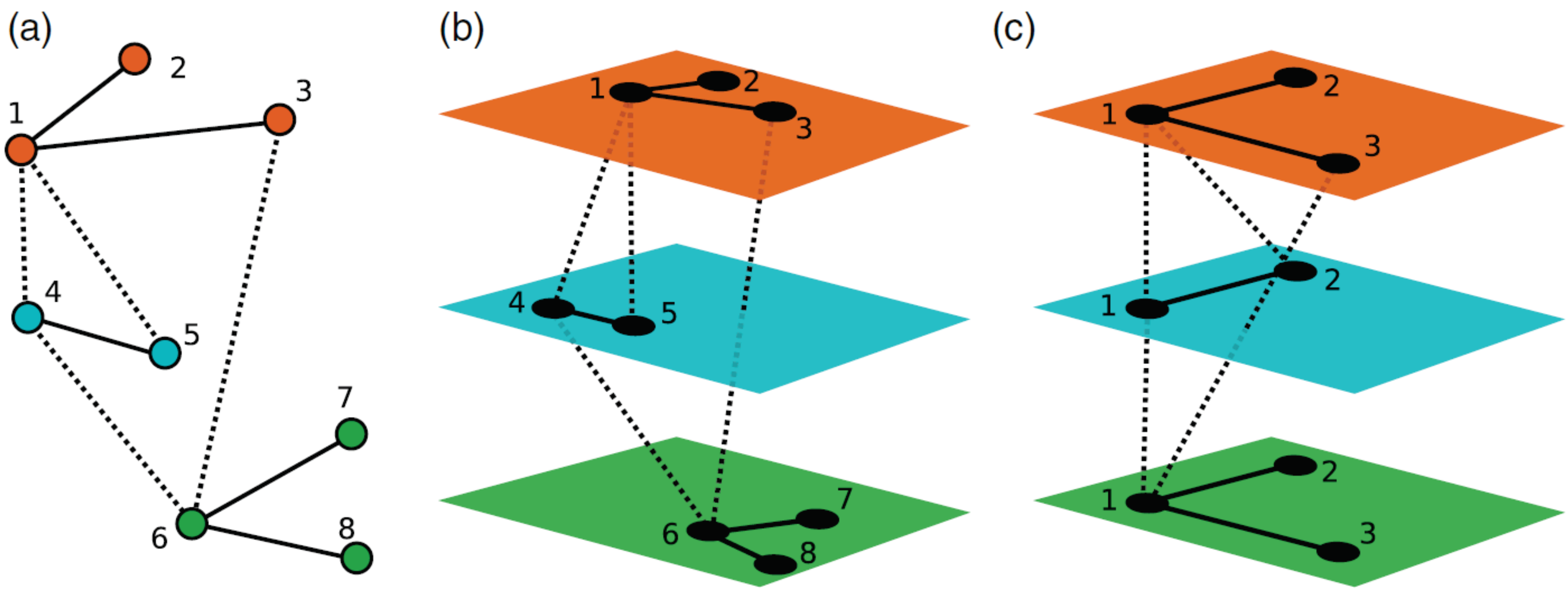
Two major subclasses of multilayer networks

- Node-colored networks
  (a.k.a. “interconnected networks”, “interdependent networks”, “networks of networks”)

- Edge-colored networks
  (a.k.a. “multiplex networks”, “multirelational networks”)

Node-colored networks

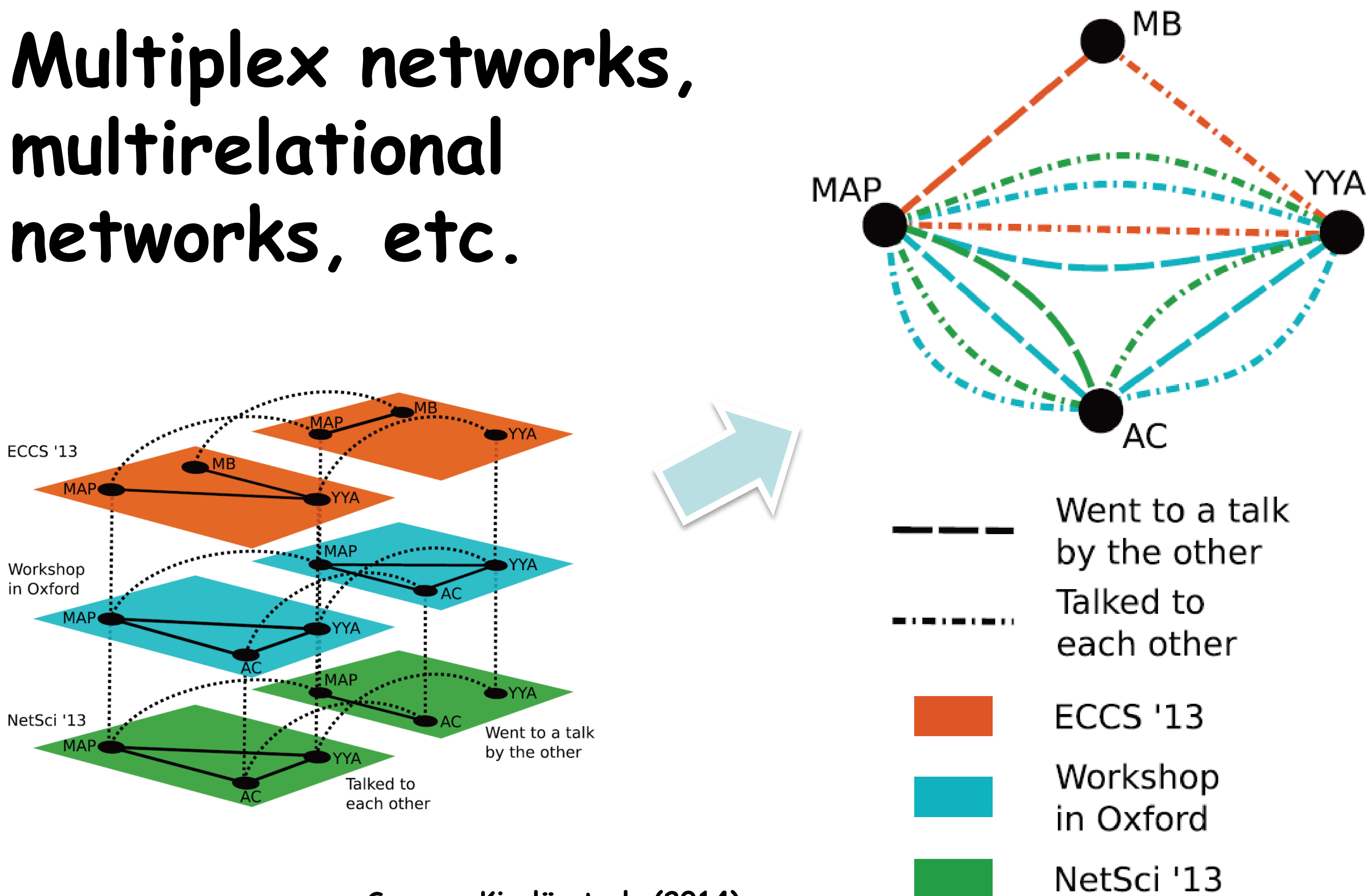
- Interconnected networks, interdependent networks, networks of networks, etc.



Source: Kivelä et al. (2014)

Edge-colored networks

- Multiplex networks, multirelational networks, etc.



Source: Kivelä et al. (2014)

Exercise

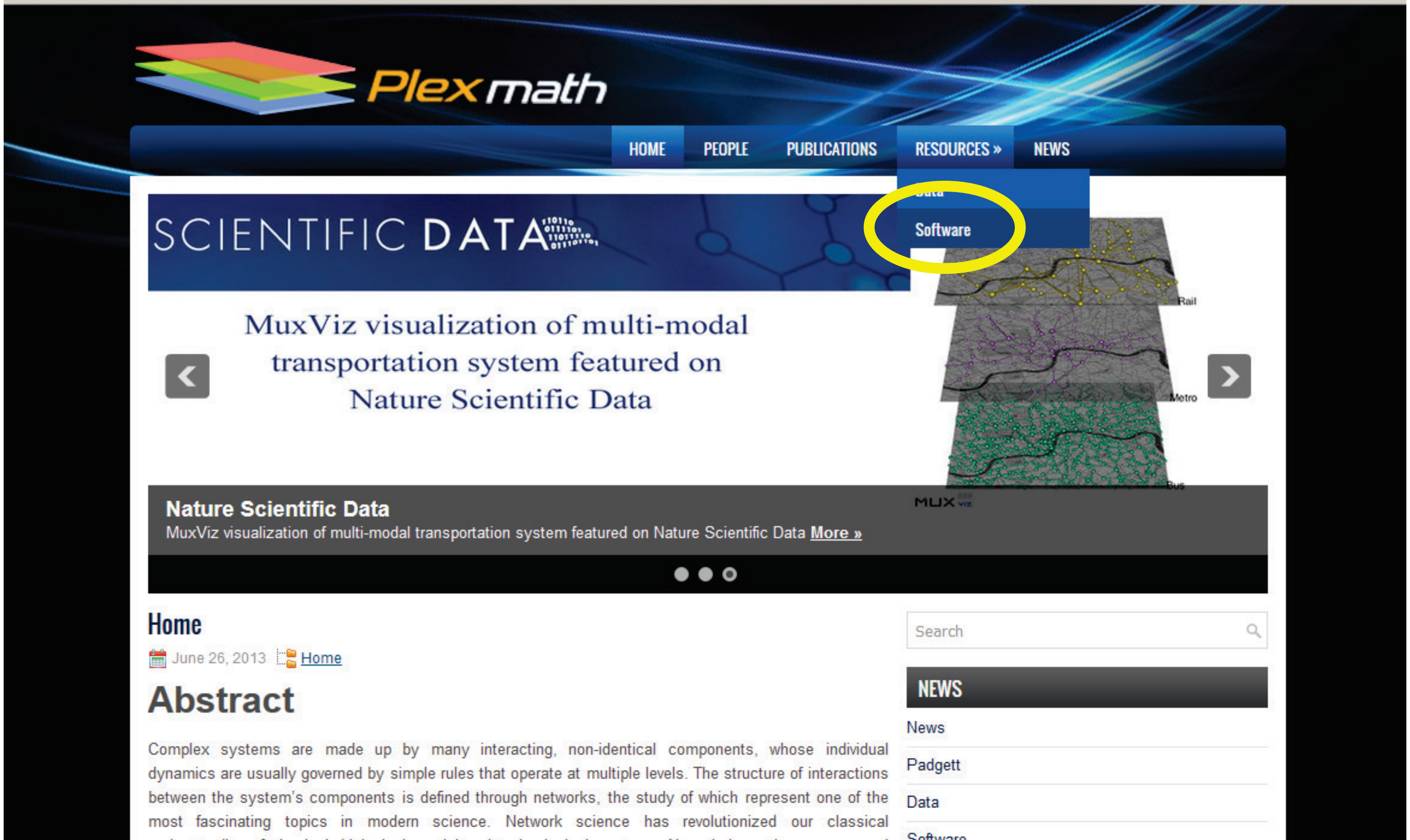
- Give a few real-world examples of (a) node-colored and (b) edge-colored multilayer networks

- Discuss properties of those networks, especially the types of their inter-layer connections

Computational Modeling and Analysis of Multilayer Networks

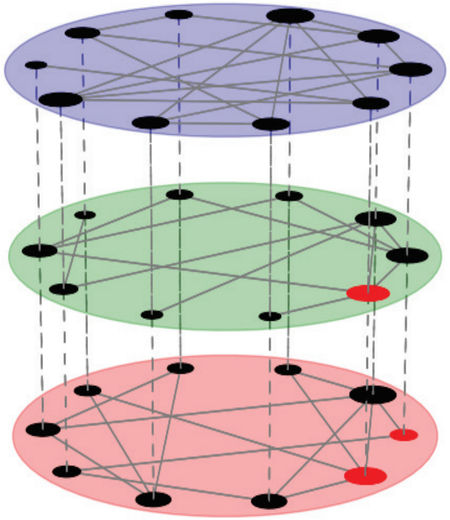
PlexMath project

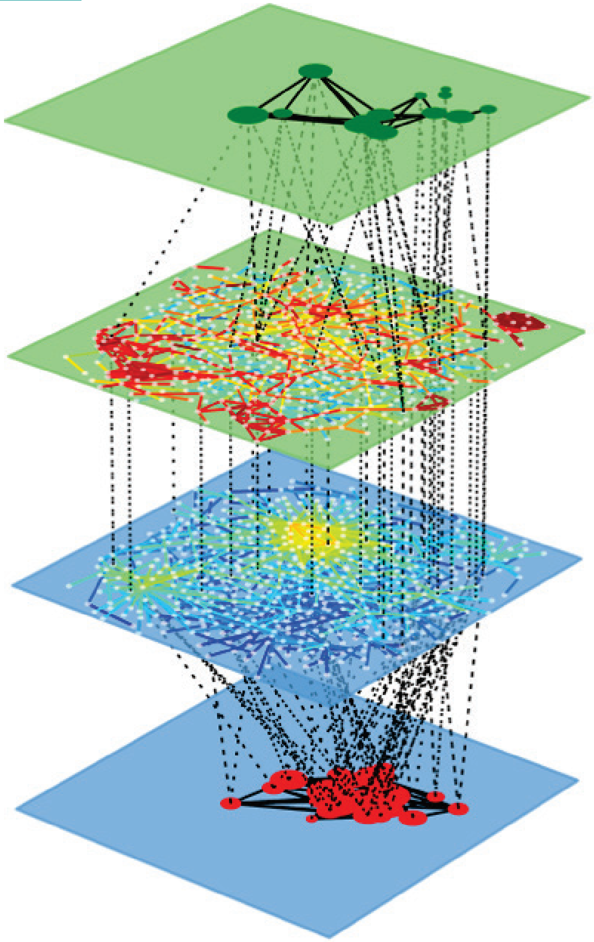
- http://www.plexmath.eu/



Pymnet library

- http://people.maths.ox.ac.uk/kivela/mln_library/
  - Developed by Mikko Kivelä





Exercise

- Download the pymnet library and its documentation
- Follow its tutorial to learn how to build multilayer/multiplex networks
  - Place the "pymnet" folder to your working directory, or add it to your Python path
  - Commands are different from NetworkX, but access to all NetworkX commands are provided under "nx." prefix

Pymnet's data structure

- MultilayerNetwork(aspects = **)
  - To represent general multilayer networks

- MultiplexNetwork(couplings = **)
  - To specifically represent multiplex networks

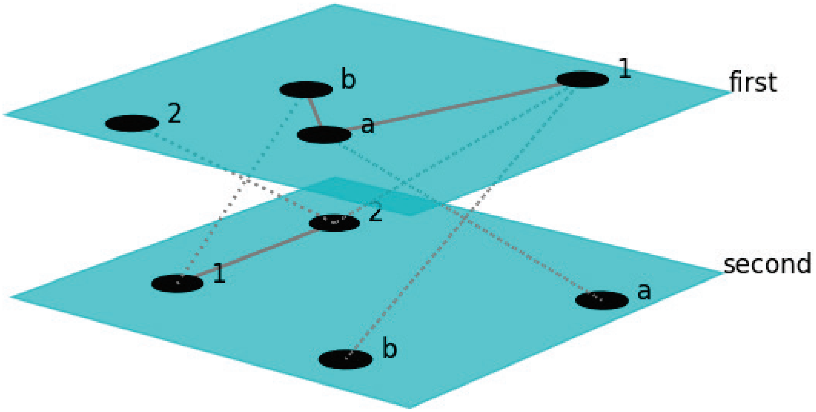
  - nx.Graph(g) converts these into a NetworkX Graph (monolayer only)

Pymnet: Basics

- add_node(), add_layer()
- g[ID][ID] = 1 adds edge (ID: sequence of indices, including aspects)
- list(g) gives node list
- list(g.iter_**()) gives node, layer, or node-layer list
- list(g.edges) gives edge list
- list(g[ID]) gives neighbor list
- g[ID].deg() or .str() gives degree or strength

Drawing multilayer networks

- Just draw(g) !!



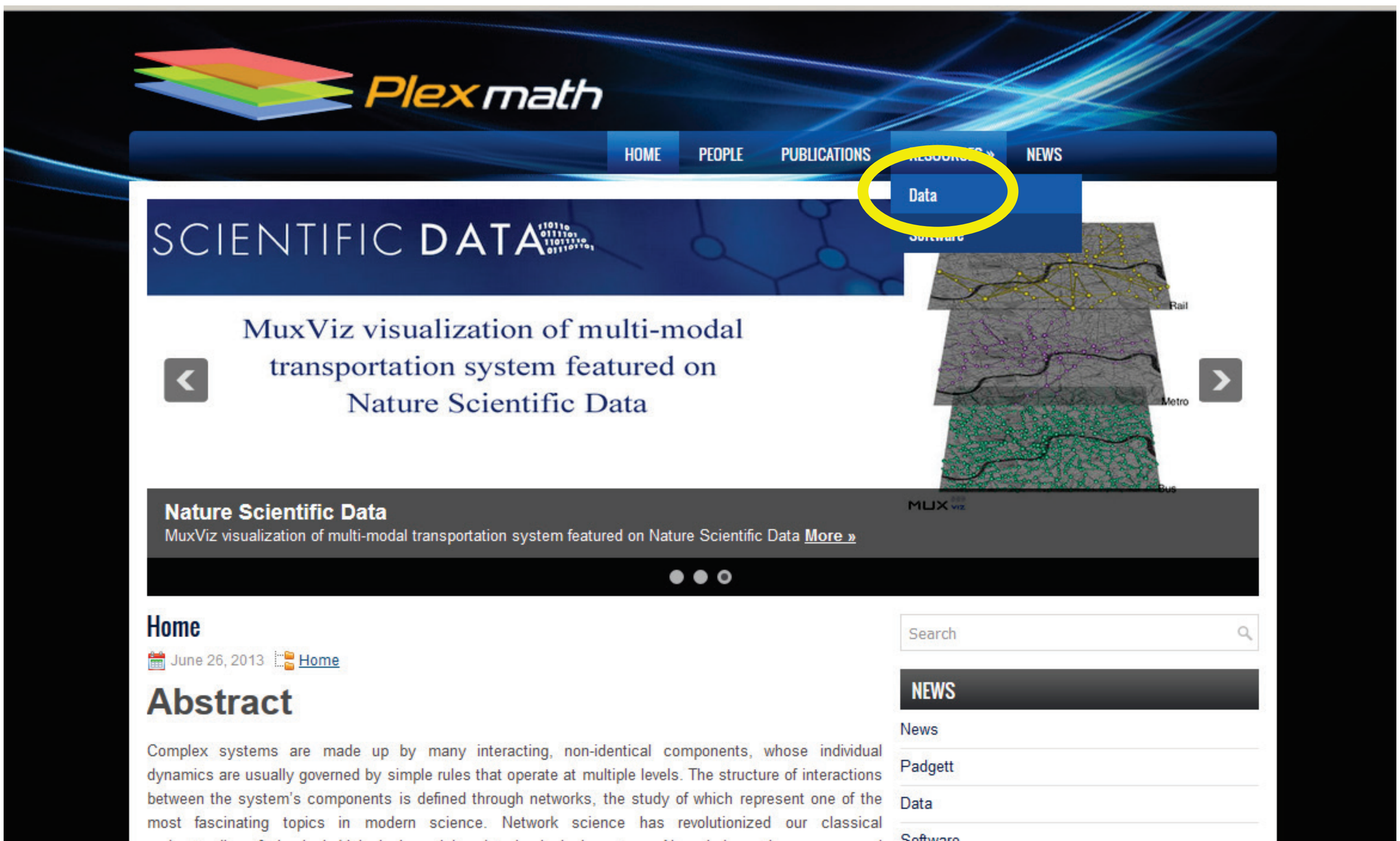
  - If you draw in a separate window, you can click and drag to rotate it
  - Many options are available to customize visualization results

Exercise

- Check out the references of pymnet
  - http://people.maths.ox.ac.uk/kivela/mln_library/reference.html#reference
- Create a few multilayer/multiplex networks and visualize them
- Explore customization of visualization as you like

Get the data!

- http://www.plexmath.eu/



Exercise

- Download the "Vickers & Chan 7th graders social network" data from the PlexMath website
- Write a Python code to read the data and construct a multilayer network
  - As a MultilayerNetwork object
  - As a MultiplexNetwork object
- Visualize it

Some built-in measurements

- degs(g) gives degree distribution
- density(g) gives network density
- multiplex_degs(g) for each layer
- multiplex_density(g) for each layer
- Various clustering measurements
  - But not so many functions available yet… (the author Mikko said he would welcome any contributions!)

Exercise

- Plot the degree distribution for each layer in the Vickers & Chan 7th graders multiplex network

- Measure the density of each layer and determine which layer was most/least dense

Other network measurements

- You can represent a multilayer network as a plain monolayer network and apply various measurements
- Is it a multiplex network and does changing layers take no cost or distance?
  - Yes to both → *Aggregate the network*
  - Otherwise → Flatten the network

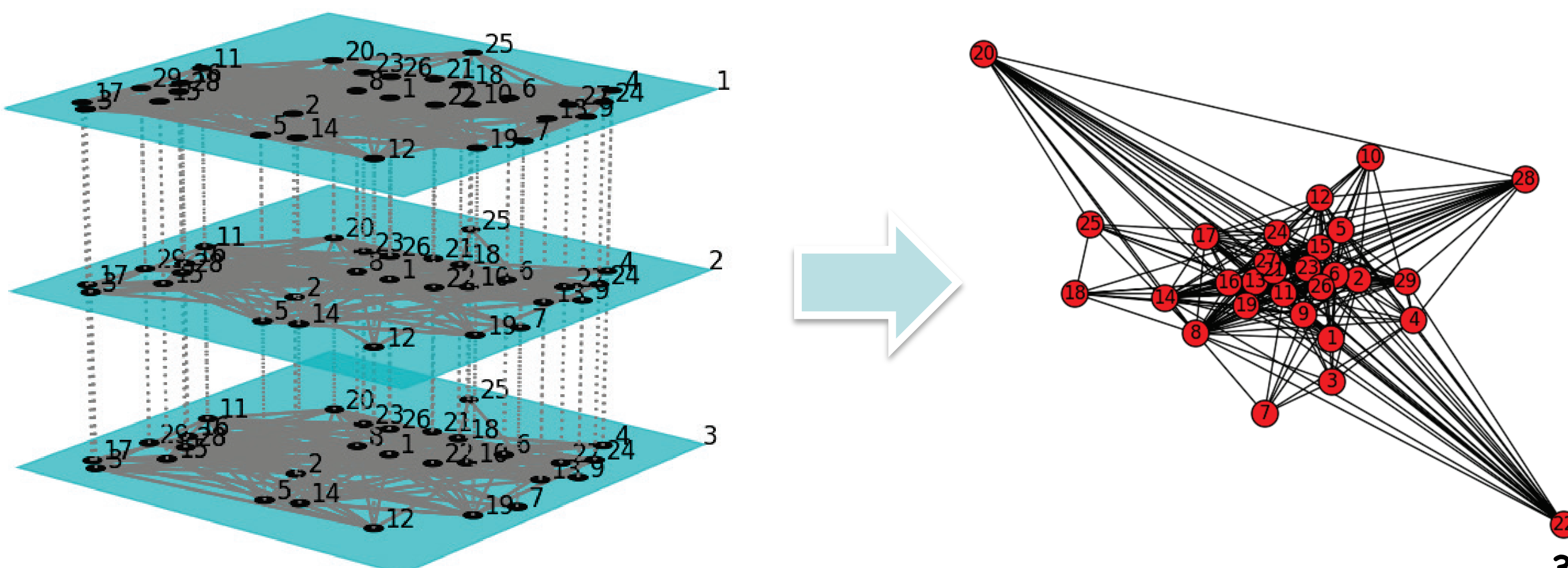
Note: When you create a monolayer network…

- Consider how the weights of intra-layer edges compare to each other between different layers
- Each layer may represent different connectivity; adjust their weights as needed
  - E.g.:

    Best friend, friend, acquaintance
    Flight, train, bus, bike, walk

Aggregating layers

- **aggregate(g, aspects)** creates a simple network aggregated over aspects
  - Works for both multilayer and multiplex, but most meaningful for multiplex

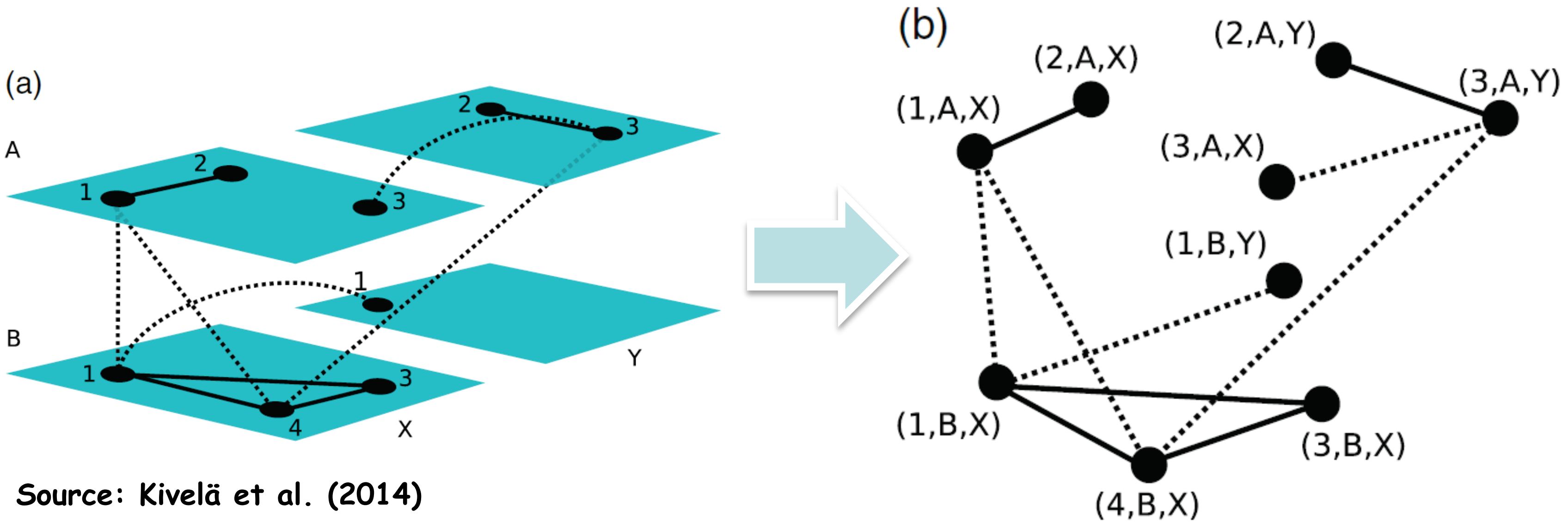


Exercise

- Create an aggregated network of the Vickers & Chan 7th graders multiplex network
- Calculate several shortest paths, average shortest path length, and several centralities of the nodes
- Detect communities using the Louvain method

"Flattening" multilayer networks

- Ignore layers and represent all the connections in a plain network

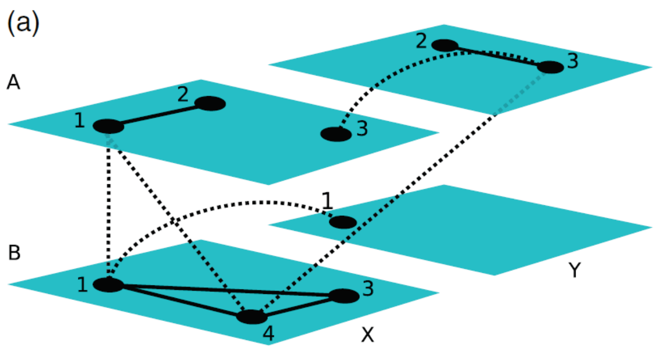


Source: Kivelä et al. (2014)

- (This is essentially the same as the construction of a space-time network for a temporal network)

Supra-adjacency matrix

- An adjacency matrix of a flattened version of a multilayer network



		A	A	B	B
		X	Y	X	Y
A	X	Adj. matrix of (A,X)	Inter-layer matrix	Inter-layer matrix	Inter-layer matrix
A	Y	Inter-layer matrix	Adj. matrix of (A,Y)	Inter-layer matrix	Inter-layer matrix
B	X	Inter-layer matrix	Inter-layer matrix	Adj. matrix of (B,X)	Inter-layer matrix
B	Y	Inter-layer matrix	Inter-layer matrix	Inter-layer matrix	Adj. matrix of (B,Y)

If inter-layer edges are "diagonal" couplings, the inter-layer matrices are diagonal too (hence the name)

Supra-adjacency matrix in pymnet

- supra_adjacency_matrix(g) creates the supra-adjacency matrix of g and a list of node IDs (arranged in the order used in the matrix)

- nx.from_numpy_matrix(A), with A being the supra-adjacency matrix, creates a flattened monolayer network

Exercise

- Create a flattened network of the Vickers & Chan 7th graders multiplex network
- Calculate several shortest paths, average shortest path length, and several centralities of the nodes
- Detect communities using the Louvain method

Inter-layer measurement

- "Interdependence"

  Ratio of the number of shortest paths that use multiple layers to the total number of shortest paths

High interdependence

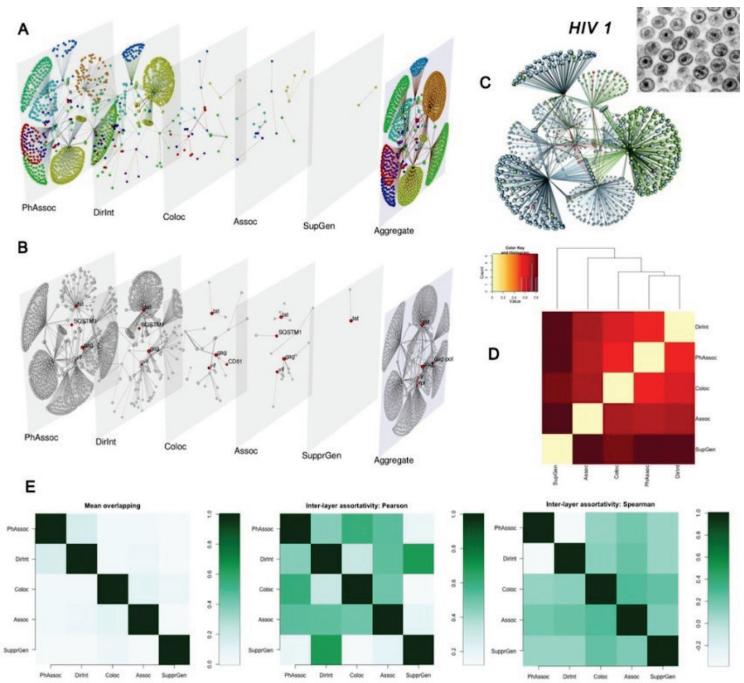
→ Communication/transportation in a multilayer network uses multiple layers more often (i.e., those layers depend on each other)

Exercise

- Design and implement a code to measure the level of interdependence in the Vickers & Chan 7th graders multiplex network
- Apply the code to other multilayer network data

FYI: If you use R…

- **muxViz**
  - http://muxviz.net
  - http://www.youtube.com/watch?v=gcpYSdi_-xI



Developed by
Manlio De Domenico

Dynamical Processes on Multilayer Networks

Dynamics on multilayer networks

- Several dynamical models have been studied recently
- Mostly focused on two-layer multiplex networks

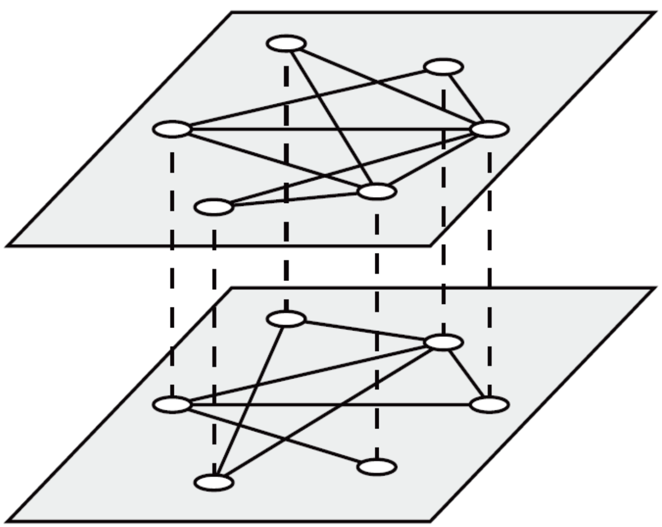
Cascade of failures

- Buldyrev, S. V., Parshani, R., Paul, G., Stanley, H. E., & Havlin, S. (2010). Catastrophic cascade of failures in interdependent networks. *Nature, 464*(7291), 1025-1028.

- Proposed a model of two-layer interdependent network (e.g., power grid and the Internet)
- Fragmentation by a cascade of failures occurs quite differently on this model than monolayer networks

Diffusion

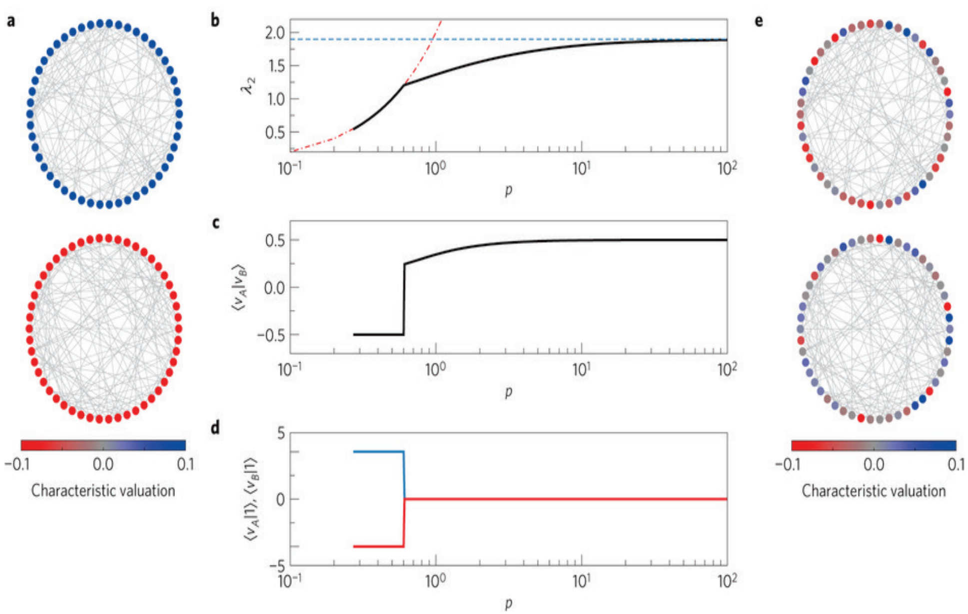
- Gomez, S., Diaz-Guilera, A., Gomez-Gardeñes, J., Perez-Vicente, C. J., Moreno, Y., & Arenas, A. (2013). Diffusion dynamics on multiplex networks. *Physical Review Letters, 110*(2), 028701.



- Studied diffusion processes on two-layer multiplex networks
- Analyzed the spectrum of a "supra-Laplacian" matrix of the network

Structural transition

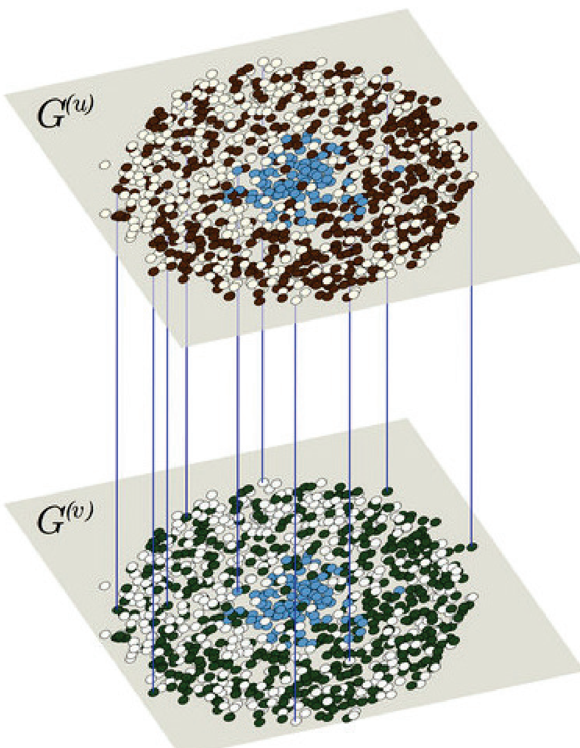
- Radicchi, F., & Arenas, A. (2013). Abrupt transition in the structural formation of interconnected networks. *Nature Physics*, *9*(11), 717-720.



- Discovered that the algebraic connectivity and Fiedler's vector undergo a sharp transition as inter-layer connectivity is increased on two-layer multiplex networks

Pattern formation

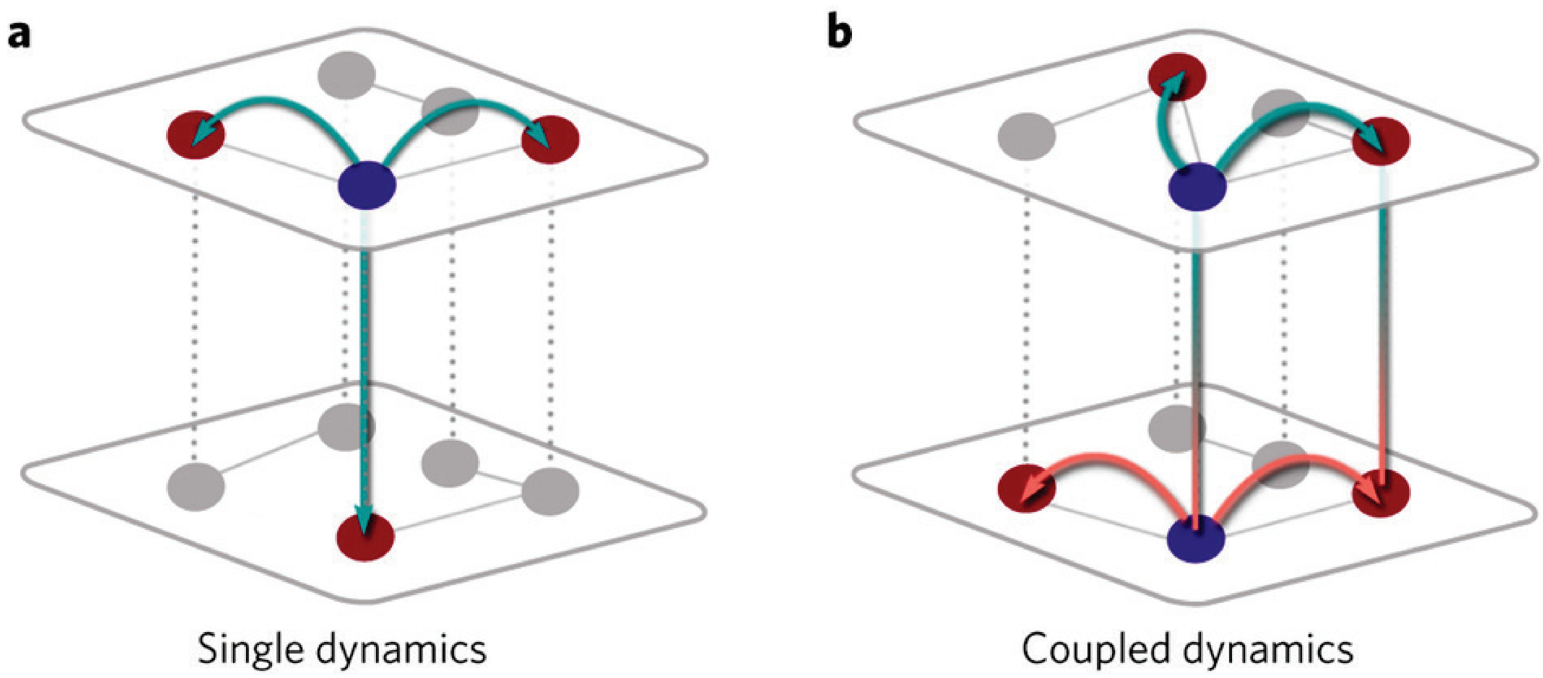
- Kouvaris, N. E., Hata, S., & Díaz-Guilera, A. (2015). Pattern formation in multiplex networks. *Scientific reports*, *5*, 10840.



- Studied reaction-diffusion dynamics on two-layer multiplex networks
- Showed that the difference in topologies between two layers can cause pattern formation (a.k.a. Turing instability) *even if diffusion constants are the same*

Spreading

- De Domenico, M., Granell, C., Porter, M. A., & Arenas, A. (2016). The physics of spreading processes in multilayer networks. *Nature Physics* 12, 901–906.



Exercise

- Implement a “cascade of failure” model on a random multilayer network with k layers
- Conduct numerical simulations with the probabilities of intra- and inter-layer connections systematically varied
- Identify the parameter values with which large-scale failure occurs